Journal of Mathematical Analysis and Applications **255,** 137–146 (2001)
doi:10.1006/jmaa.2000.7209, available online at http://www.idealibrary.com on IDEAL®

Estimates of the Sobolev Norm of a Product of Two Functions

Josip Tambača

Department of Mathematics, University of Zagreb, Bijenička 30, 10 000 Zagreb, Croatia
E-mail: tambaca@math.hr

Submitted by B. Bongiorno

Received January 4, 2000

In this paper we estimate the Sobolev norm of a product of two scalar functions. The proof is direct and it follows by use of the Littlewood–Paley theory. © 2001 Academic Press

Key Words: the term $u\nabla v$; the Sobolev spaces; Littlewood–Paley theory.

1. INTRODUCTION

In this paper we consider the best possible asymmetric estimate of a product of two scalar functions defined on $\mathbb{R}^n$. The estimate is particularly interesting for products of the form $u\nabla v$ which appear in the Navier–Stokes equation and various kinetic equations as well. The results similar to the one presented here can be found in [6, 7]; however, our proofs are different. Namely, we phrase the statement in terms of the Bessel spaces and prove it directly by use of the Lizorkin–Triebel spaces and the Littlewood–Paley theory. We follow ideas presented in the paper [3] extending the result therein on products of functions for a greater class of function spaces. The proof of Theorem 1.2 in [3] is complete for $d \geq 3$ and we follow it, while the cases $d = 0, 1, 2$ require a different argument which we give here. Correcting this point also requires that the statement of the main theorem in [3] must be modified (see Corollary 2.2).

By $\mathscr{S}$ we denote the Schwartz class of rapidly decreasing C^∞-functions on $\mathbb{R}^n$. Its topological dual, the space of tempered distributions, is denoted by $\mathscr{S}'$; the space of compactly supported C^∞-functions on $\mathbb{R}^n$ we denote by $\mathscr{D}$, while $\mathscr{O}$ denotes the space of polynomially bounded C^∞-functions on $\mathbb{R}^n$. The Fourier transform of a tempered distribution f is denoted by $\mathscr{F}f$.

0022-247X/01 $35.00

For $f \in \mathscr{S}$, it is defined by

$$(\mathscr{F}f)(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi} f(x)\, dx.$$

For $z \in \mathbb{R}$, the Bessel kernel is given by $G_z = \mathscr{F}^{-1}(1 + 4\pi^2|\xi|^2)^{-z/2}$. If $z > 0$, the function G_z is positive, integrable, radial, and radially decreasing. For $z \in \mathbb{R}$, it holds $\mathscr{F}G_z \in \mathscr{O}$, so the operator

$$\mathscr{G}_z : \mathscr{S}' \to \mathscr{S}', \qquad \mathscr{G}_z g = G_z * g$$

is well-defined and injective. For $1 \le p < \infty$ and $z \in \mathbb{R}$, we define the Bessel potential spaces $L^{p,z}$ by $L^{p,z} = L^{p,z}(\mathbb{R}^n) = \mathscr{G}_z L^p(\mathbb{R}^n)$. Then the map $\mathscr{G}_z|_{L^p} : L^p \to L^{p,z}$ is invertible with inverse $\mathscr{G}_{-z}$. Equipped with the norm $\|f\|_{p,z} = \|g\|_p = \|\mathscr{F}^{-1}((1 + 4\pi^2|\cdot|^2)^{z/2}\mathscr{F}f)\|_p$, where $\|\cdot\|_p$ is the usual L^p-norm, $L^{p,z}$ becomes a Banach space with dual space $L^{q,-z}$, where $1/p + 1/q = 1$. All dual products we denote by $\langle\cdot,\cdot\rangle$. The Schwartz class, the space $\mathscr{D}$, and the set $\mathscr{P} = \{f \in \mathscr{S} : \mathscr{F}f \in \mathscr{D}\}$ are dense in $L^{p,z}$ for any $1 \le p < \infty$, $z \in \mathbb{R}$. If $s < r$ and $1 \le p < \infty$ then $L^{p,r}$ is continuously imbedded in $L^{p,s}$. We make use of the following fundamental result: $L^{p,z}$ coincides with the Sobolev space $W^{z,p}(\mathbb{R}^n)$ for integer values of z if $1 < p < \infty$, and for all $z \in \mathbb{R}$ when $p = 2$. In general some imbeddings hold (see [1, 2]).

2. THE INEQUALITY

Now we are able to state the main result of the paper:

THEOREM 2.1. *Let $1 < p < \infty$ and $r, s \in \mathbb{R}$, $n \in \mathbb{N}$ such that*

$$r < \frac{n}{p}, \qquad s < \frac{n}{p}, \qquad r + s > \max\left\{-\frac{n}{q} + \frac{n}{p}, 0\right\},$$

where $1/p + 1/q = 1$. Then for $t = r + s - n/p$ and distributions $u \in L^{p,s}$, $v \in L^{p,r}$ such that if $p > 2$ one of the following is fulfilled: $u \in L^{q,-r}$ or $v \in L^{q,-s}$, one has $uv \in L^{p,t}$ and the following inequality holds

$$\|uv\|_{p,t} \le C\|u\|_{p,s}\|v\|_{p,r}.$$

Note that the conditions of the theorem imply $r + s > 0$. Therefore at least one of the distributions $u \in L^{p,s}$, $v \in L^{p,r}$ is actually a function. A product of a function and a distribution is not defined in general, so the additional assumption, $u \in L^{q,-r}$ or $v \in L^{q,-s}$, allows us to give the meaning to the term uv via the so-called duality method [4]. Let us assume that $u \in L^{q,-r}$. Then there is a unique distribution χ such that

$$_{\mathscr{D}'}\langle \chi, \varphi \rangle_{\mathscr{D}} = {}_{L^{p,r}}\langle v, u\varphi \rangle_{L^{q,-r}}, \qquad \varphi \in \mathscr{D}.$$

Therefore we denote $\chi = uv$, which has the usual meaning when u and v are functions. If $1 < p \le 2$, then $1 \le p \le q = p/(p-1) < np/(n-(s+r)p)$ by assumptions of Theorem 2.1. Hence $L^{p,s}$ is continuously imbedded in $L^{q,-r}$, as well as $L^{p,r}$ in $L^{q,-s}$ (see [1, Theorem 7.63]). Therefore the additional assumption is unnecessary and we omit it in the theorem. For $p = 2$ because of $s + r > 0$ it follows $L^{2,s} \subset L^{2,-r}$ and $L^{2,r} \subset L^{2,-s}$ and the conclusion easily follows.

We stress the special case $p = 2$ and specialize Theorem 2.1 for the product $u\nabla v$ and $r = s$:

COROLLARY 2.1. *Let $n \in \mathbb{N}$, $p = 2$, and $r, s \in \mathbb{R}$ such that $r < \frac{n}{2}$, $s < \frac{n}{2}$, $0 < r + s$. Then for $t = r + s - n/2$ and distributions $u \in L^{2,s}$, $v \in L^{2,r}$ one has $uv \in L^{2,t}$ and the following inequality holds*

$$\|uv\|_{2,t} \le C\|u\|_{2,s}\|v\|_{2,r}.$$

COROLLARY 2.2. *Let $1 < p < \infty$ and $t \in \mathbb{R}$, $n \in \mathbb{N}$ such that*

$$\max\left\{-\frac{n}{q}, -\frac{n}{p}\right\} < t < \frac{n}{p} - 1,$$

where $1/p + 1/q = 1$. Then for $s = 1/2(n/p + t + 1)$ and distributions u, $v \in L^{p,s}$ such that if $p > 2$ one of the following is fulfilled, $u \in L^{q,-s+1}$ or $v \in L^{q,-s+1}$, one has $u\nabla v \in L^{p,t}$ and the following inequality holds

$$\|u\nabla v\|_{p,t} \le C\|u\|_{p,s}\|v\|_{p,s}.$$

3. THE PROOF

Let $\phi \in \mathscr{S}$, supp $\mathscr{F}\phi \subset B(0,1)$, and $\mathscr{F}\phi = 1$ on $B(0,\frac{1}{2})$. We define

$$\phi_k(x) = 2^{kn}\phi(2^k x),\ k \in \mathbb{N} \cup \{0\},$$
$$\psi_k(x) = \phi_k(x) - \phi_{k-1}(x),\ k \in \mathbb{N}.$$

Then it is easy to show

$$\begin{aligned} &\operatorname{supp}\mathscr{F}\phi_k \subset B(0,2^k), \qquad k \in \mathbb{N} \cup \{0\}, \\ &\operatorname{supp}\mathscr{F}\psi_k \subset B(0,2^k) \setminus B(0,2^{k-2}), \qquad k \in \mathbb{N}. \end{aligned} \tag{1}$$

To shorten notations we denote $\psi_0 = \phi$ and $\psi_k'(x) = \psi_k(-x)$. We define the following linear operators on $\mathcal{S}'$ by

$$S_k f = \phi_k * f, \qquad \Delta_k f = \psi_k * f, \qquad \Delta'_k f = \psi'_k * f,\ k \in \mathbb{N} \cup \{0\}.$$

The following identities hold

$$\sum_{k=0}^{\infty} \Delta_k = I, \qquad \sum_{k=0}^{\infty} \Delta'_k = I, \tag{2}$$

where the convergence of the sums is strong in any of the spaces $\mathcal{S}, \mathcal{S}', L^{p,z}$.

The Lizorkin–Triebel space $F_z^{p,\alpha}$ is a space of all $v \in \mathcal{S}'$ such that

$$\|v\|_{F_z^{p,\alpha}} = \left\| \left(\sum_{k=0}^{\infty} 2^{\alpha z k} |\psi_k * v|^{\alpha} \right)^{1/\alpha} \right\|_p < \infty.$$

The proof of Theorem 2.1 is based on the fact that $F_z^{p,2}$ coincides with $L^{p,z}$ for any $z \in \mathbb{R}$ and $1 < p < \infty$ with equivalence of norms [2, Theorem 4.2.2]; i.e., there are constants A_1, A_2 such that

$$\forall v \in L^{p,z} \qquad A_1 \|v\|_{p,z} \le \|v\|_{F_z^{p,2}} \le A_2 \|v\|_{p,z}. \tag{3}$$

In analogy with [5, VI 7.14] we state:

LEMMA 3.1. *Let $0 < a \le b < \infty$ and $1 < p < \infty$, $z \in \mathbb{R}$. Suppose that a family of functions f_k, $k \in \mathbb{N}$ is such that $(2^{zk} f_k)_k \in L^p(l^2)$ and $\mathcal{F} f_k$ is supported in $\{\xi : a2^k \le |\xi| \le b2^k\}$. Then*

$$\left\| \sum_{k=0}^{\infty} f_k \right\|_{p,z} \le A \left\| \left(\sum_{k=0}^{\infty} 2^{2zk} |f_k|^2 \right)^{1/2} \right\|_p .$$

Proof. For a fixed $l \in \mathbb{N}$ let us define the operator $\overline{\Delta}{}^l_k g = \sum_{|j| \le l} \Delta_{k-j} g$. Here we assume that $\Delta_k = 0$ if $k < 0$. Then by the Minkowski inequality and the equivalence of the norms (3) it follows

$$\begin{aligned} &\left\| \left(\sum_{k=0}^{\infty} 2^{-2zk} |\overline{\Delta}{}^l_k g|^2 \right)^{1/2} \right\|_q \\ &\qquad \le (2l+1) \left\| \left(\sum_{k=0}^{\infty} 2^{-2zk} |\Delta_k g|^2 \right)^{1/2} \right\|_q \le C \|g\|_{q,-z}. \end{aligned} \tag{4}$$

Let l be chosen such that $1/a < 2^l$, $b < 2^{l-2}$. Then for fixed k and $j \in \mathbb{N}$ such that $j > l$ it follows $b2^k < 2^{k+j-2}$, so $\operatorname{supp} \mathscr{F}\psi_{k+j} \cap \operatorname{supp} \mathscr{F} f_k = \emptyset$. In the same way for $-j < -l$ one gets $2^{k-j} < a2^k$, so $\operatorname{supp} \mathscr{F}\psi_{k-j} \cap \operatorname{supp} \mathscr{F} f_k = \emptyset$. Therefore $f_k = \sum_{i=0}^{\infty} \Delta'_i f_k = \sum_{i=k-l}^{k+l} \Delta'_i f_k = \overline{\Delta}'^l_k f_k$. Hence, for any function $g \in \mathscr{P}$ we calculate

$$\left\langle \sum_{k=0}^{\infty} f_k, g \right\rangle = \sum_{k=0}^{\infty} \langle \overline{\Delta'}^l_k f_k, g \rangle = \sum_{k=0}^{\infty} \langle f_k, \overline{\Delta}^l_k g \rangle,$$

where the sums are finite by the definition of $\mathscr{P}$. It follows that

$$\left| \left\langle \sum_{k=0}^{\infty} f_k, g \right\rangle \right| \le C \left\| \left(\sum_{k=0}^{\infty} 2^{2kz} |f_k|^2 \right)^{1/2} \right\|_p \|g\|_{q,-z}$$

by the Hölder inequality applied twice and the inequality (4), where $1/p + 1/q = 1$. This proves the lemma by density of $\mathscr{P}$ in $L^{q,-z}$. ∎

Let us denote $\tilde{\psi}_k(x) = |\psi_k(x)|$. As a consequence of the Feffermann–Stein maximal inequality [2, Theorem 1.1.2] one has:

LEMMA 3.2. *Suppose* $1 < p < \infty$ *and* $1 < \alpha < \infty$. *Then there is a constant* A *such that*

$$\left\| \left(\sum_{k=0}^{\infty} |\tilde{\psi}_k * f_k|^{\alpha} \right)^{1/\alpha} \right\|_p \le A \left\| \left(\sum_{k=0}^{\infty} |f_k|^{\alpha} \right)^{1/\alpha} \right\|_p,$$

for all sequences $(f_k)_k \in L^p(l^{\alpha})$ *on* $\mathbb{R}^n$.

Proof of Theorem 2.1. In the sequel p and q denote conjugate indices $1/p + 1/q = 1$. Let $u \in L^{p,s} \cap L^{q,-r}$, $v \in L^{p,r}$ (the same argument applies for the other assumption). If $r > 0$ and $\varphi \in \mathscr{S}$ by positivity of the Bessel kernel G_r it follows

$$\|u\varphi\|_{q,-r} = \|G_r * (u\varphi)\|_q \le \|\varphi\|_{\infty} \|G_r * |u|\,\|_q \le \|\varphi\|_{\infty} \|u\|_{q,-r}. \tag{5}$$

By duality the result follows for $r < 0$ as well. Therefore there is unique $\chi \in \mathscr{S}'$ such that

$${}_{\mathscr{S}'}\langle \chi, \varphi \rangle_{\mathscr{S}} = {}_{L^{p,r}}\langle v, u\varphi \rangle_{L^{q,-r}}, \qquad \varphi \in \mathscr{S}$$

and defines the product $uv \in \mathscr{S}'$ by means of the duality method [4].

In the rest of the proof we estimate the product uv. We prove the inequality for u and v in a dense subset and then extend the inequality by density.

The convolution of any tempered distribution with ψ_j is denoted by index j ($v_j = \psi_j * v$); we adopt the convention that $v_l = 0$ for negative l. In general $v \in \mathscr{S}'$ and $\psi_j \in \mathscr{S}$. Hence $v_j \in \mathscr{O}$ and $\mathscr{F}v_j = \mathscr{F}\psi_j \mathscr{F}v$.

The following inequalities will be used

$$\|\psi_j\|_\infty \leq C2^{nj}, \tag{6}$$

$$\|\psi_j\|_{q,-z} \leq C2^{(n/p-z)j}, \tag{7}$$

$$\|u_j\|_\infty \leq 2^{(n/p-z)j}\|u\|_{p,z}, \tag{8}$$

$$\|u_j\|_p \leq C2^{-zj}\|u\|_{p,z}, \tag{9}$$

for any $u \in L^{p,z}$, $1 < p < \infty$, $z \in \mathbb{R}$. The estimate (6) is a direct consequence of the definition of ψ_j, while the estimate (7) uses Lemma 3.1 and $\|\psi_j\|_q = 2^{(n/p)j}\|\phi(\cdot) - 2^{-n}\phi(\cdot/2)\|_q$. For (8) we argue as

$$\begin{aligned}|u_j(x)| &= |\langle u, \psi_j(x-\cdot)\rangle| \leq \|u\|_{p,z}\|\psi_j(x-\cdot)\|_{q,-z} \\ &\leq \|u\|_{p,z}\|\psi_j\|_{q,-z} \leq C2^{(n/p-z)j}\|u\|_{p,z},\end{aligned}$$

by translation invariance in the argument of the $L^{q,-z}$ norm. The inequality (9) is a direct consequence of (3).

Let $u, v \in \mathscr{O}$ such that $\mathscr{F}u$, $\mathscr{F}v$ have compact support. Equation (2) implies

$$u(x)v(x) = \sum_{\substack{j=0\\l=0}}^{\infty} u_j(x)v_l(x) = \sum_{\substack{d=0\\j=0}}^{\infty} u_{j+d}(x)v_j(x) + \sum_{\substack{d=1\\j=0}}^{\infty} u_j(x)v_{j+d}(x), \tag{10}$$

where the sums are finite.

Fix $d > 2$ and consider the first term in the right hand side of (10). Because of $\operatorname{supp}\mathscr{F}(u_{j+d}v_j) \subset \{\xi : 2^{j+d}(2^{-2} - 2^{-d}) \leq |\xi| \leq 2^{j+d}(1 + 2^{-d})\}$ the assumptions of Lemma 3.1 are fulfilled. Hence

$$\left\|\sum_{j=0}^{\infty} u_{j+d}v_j\right\|_{p,t} \leq C\left\|\left[\sum_{j=0}^{\infty} 2^{2t(j+d)}|u_{j+d}v_j|^2\right]^{1/2}\right\|_p.$$

The L^∞ estimate (8) applied on v_j implies

$$\begin{aligned}\left\|\sum_{j=0}^{\infty} u_{j+d}v_j\right\|_{p,t} &\leq C2^{-(s-t)d}\|v\|_{p,r}\left\|\left[\sum_{j=0}^{\infty} 2^{2(j+d)s}|u_{j+d}|^2\right]^{1/2}\right\|_p \\ &\leq C2^{-(s-t)d}\|u\|_{p,s}\|v\|_{p,r}.\end{aligned}$$

Because $s - t = n/p - r > 0$, the last estimate is summable over d.

For the second sum in (10) and $d > 2$ we argue the same way, but estimating u_j in L^∞ norm. We obtain the estimate

$$\left\| \sum_{j=0}^{\infty} u_j v_{j+d} \right\|_{p,t} \leq C 2^{-(r-t)d} \|u\|_{p,s} \|v\|_{p,r}.$$

Because $r - t = n/p - s > 0$, this estimate is summable over d as well.

The cases $d = 0, 1, 2$, in the first sum in (10), need separate treatment. We use (3) to obtain

$$\begin{aligned} \left\| \sum_{j=0}^{\infty} u_{j+d} v_j \right\|_{p,t} &\leq C \left\| \left[\sum_{k=0}^{\infty} 2^{2tk} \left| \sum_{j=0}^{\infty} \psi_k * (u_{j+d} v_j) \right|^2 \right]^{1/2} \right\|_p \\ &\leq C \left\| \left\| \left(\sum_{j=0}^{\infty} |2^{tk} \psi_k * (u_{j+d} v_j)| \right)_k \right\|_{l^2} \right\|_p . \end{aligned}$$

The term $u_{j+d} v_j$ is supported in frequency on $|\xi| \leq 2^j(2^d + 1)$. Hence the terms $\psi_k * (u_{j+d} v_j)$ are identically zero for $0 \leq j \leq k - 3 - d$. Therefore

$$\begin{aligned} \left\| \sum_{j=0}^{\infty} u_{j+d} v_j \right\|_{p,t} &\leq C \left\| \left\| \left(\sum_{j=k-d-2}^{\infty} |2^{tk} \psi_k * (u_{j+d} v_j)| \right)_k \right\|_{l^2} \right\|_p \\ &\leq C \left\| \left\| \sum_{\delta=-2-d}^{\infty} \left(2^{tk} \psi_k * (u_{k+d+\delta} v_{k+\delta}) | \right)_k \right\|_{l^2} \right\|_p \\ &\leq C \left\| \sum_{\delta=-2-d}^{\infty} \| \left(2^{tk} \psi_k * (u_{k+d+\delta} v_{k+\delta}) | \right)_k \|_{l^2} \right\|_p \\ &\leq C \sum_{\delta=-2-d}^{\infty} \left\| \left[\sum_{k=0}^{\infty} 2^{2tk} |\psi_k * (u_{k+d+\delta} v_{k+\delta})|^2 \right]^{1/2} \right\|_p . \end{aligned}$$

In the sequel we show that the summands in the last sum decrease geometrically in δ; we show that for all $\delta > -3 - d$, and some $\epsilon > 0$

$$\left\| \left[\sum_{k=0}^{\infty} 2^{2tk} | \psi_k * (u_{k+\delta+d} v_{k+\delta}) |^2 \right]^{1/2} \right\|_p \leq C 2^{-\epsilon\delta} \|u\|_{p,s} \|v\|_{p,r}. \quad (11)$$

Let $1 < a,\ b < \infty$ such that $1/a + 1/b = 1$, $1 < a < p \le b < \infty$ and $a < 2$. The Hölder inequality implies

$$2^{tk}|\psi_k * (u_{k+\delta+d}v_{k+\delta})(x)|$$
$$\le 2^{tk}\Big[\tilde{\psi}_k * |u_{k+\delta+d}|^a(x)\Big]^{1/a}\Big[\tilde{\psi}_k * |v_{k+\delta}|^b(x)\Big]^{1/b}$$

$$2^{tk}\Big[\tilde{\psi}_k * |v_{k+\delta}|^b(x)\Big]^{1/b} \le 2^{tk}\|\psi_k\|_\infty^{1/b}\|v_{k+\delta}\|_p^{p/b}\|v_{k+\delta}\|_\infty^{(1-p/b)}$$
$$= C2^{s(k+\delta)-(n/b+t)\delta}\|v\|_{p,r},$$

where we also used (6), (8), and (9). It follows that

$$2^{tk}|\psi_k * (u_{k+\delta+d}v_{k+\delta})| \le C2^{s(k+\delta)-(n/b+t)\delta}\Big[\tilde{\psi}_k * |u_{k+\delta+d}|^a\Big]^{1/a}\|v\|_{p,r}. \tag{12}$$

Now, from (12) and Lemma 3.2 it follows that

$$\left\|\left[\sum_{k=0}^{\infty} 2^{2tk}|\psi_k * (u_{k+\delta+d}v_{k+\delta})|^2\right]^{1/2}\right\|_p$$
$$\le C2^{-(n/b+t)\delta-sd}\left\|\left[\sum_{k=0}^{\infty} 2^{2s(k+\delta+d)}\Big(\tilde{\psi}_k * |u_{k+\delta+d}|^a\Big)^{2/a}\right]^{1/2}\right\|_p\|v\|_{p,r}$$
$$= C2^{-(n/b+t)\delta-sd}\left\|\left[\sum_{k=0}^{\infty} 2^{2s(k+\delta+d)}\Big(\tilde{\psi}_k * |u_{k+\delta+d}|^a\Big)^{2/a}\right]^{a/2}\right\|_{p/a}^{1/a}\|v\|_{p,r}$$
$$\le C2^{-(n/b+t)\delta-sd}\left\|\left[\sum_{k=0}^{\infty} 2^{2s(k+\delta+d)}\Big(|u_{k+\delta+d}|^a\Big)^{2/a}\right]^{a/2}\right\|_{p/a}^{1/a}\|v\|_{p,r}$$
$$= C2^{-(n/b+t)\delta-sd}\left\|\left[\sum_{k=0}^{\infty} 2^{2s(k+\delta+d)}|u_{k+\delta+d}|^2\right]^{1/2}\right\|_p\|v\|_{p,r}$$
$$\le C2^{-(n/b+t)\delta-sd}\|u\|_{p,s}\|v\|_{p,r}.$$

For the last term to be summable over δ it is sufficient that

$$\frac{n}{b} + t > 0, \qquad 1 < a < p \le b < \infty, \qquad 1/a + 1/b = 1, \qquad a < 2. \tag{13}$$

For the second sum in (10) and $d = 1, 2$, we argue the same way changing the roles of u and v and s and r. We obtain the estimate

$$\left\|\left[\sum_{k=0}^{\infty} 2^{2tk}|\psi_k * (u_{k+\delta}v_{k+\delta+d})|^2\right]^{1/2}\right\|_p \le C2^{-(n/b+t)\delta-rd}\|u\|_{p,s}\|v\|_{p,r}.$$

Again, (13) is sufficient for the summability over δ of the last estimate.

For $2 \le p < \infty$ because of $r + s > 0$ it follows $p < -n/t$, so b can be chosen such that $2 \le p < b < -n/t$ and (13) follows. For $1 < p < 2$ because of condition $r + s > n/p - n/q$ it follows $q < -n/t$, so b can be chosen such that $2 < q < b < -n/t$. Again (13) is fulfilled. This proves (11).

Thus for $u, v \in \mathscr{O}$ such that $\mathscr{F}u$, $\mathscr{F}v$ are compactly supported the theorem is proved.

Let us now consider $u \in L^{p,s} \cap L^{q,-r}$, $v \in L^{p,r}$. Then $S_k u$, $S_m v \in \mathscr{O}$, $S_k u \in L^{p,s} \cap L^{q,-r}$, $S_m v \in L^{p,r}$. Moreover, $\mathscr{F}(S_k u)$ and $\mathscr{F}(S_k v)$ are compactly supported and one has the strong convergences

$$S_k u \to u, \qquad \text{in } L^{p,s}, \tag{14}$$

$$S_k u \to u, \qquad \text{in } L^{q,-r}, \tag{15}$$

$$S_m v \to v, \qquad \text{in } L^{p,r}. \tag{16}$$

Moreover, there is an uniform bound for the sequence $S_k u$

$$\|S_k u\|_{p,s} = \|\phi_k * u\|_{p,s} = \|\phi_k * (G_{-s} * u)\|_p \le \|\phi_k\|_1 \|u\|_{p,s} = \|\phi\|_1 \|u\|_{p,s}.$$

Hence for any $k, m \in \mathbb{N}$ we obtain the estimate

$$\|S_k u\, S_m v\|_{p,t} \le C \|S_k u\|_{p,s} \|S_m v\|_{p,r} \le C \|u\|_{p,s} \|v\|_{p,r}. \tag{17}$$

For fixed m, from (17), it follows that there is $\chi_m \in L^{p,t}$ such that

$$S_k u\, S_m v \to \chi_m, \qquad \text{in } L^{p,t} \text{ weakly}, \tag{18}$$

when k goes to infinity. The function $S_m v$ is in $\mathscr{O}$, so the product $u\, S_m v$ is well-defined. For $\varphi \in \mathscr{D}$ we calculate

$$\langle u\, S_m - \chi_m, \varphi \rangle = \langle u - S_k u, \varphi S_m v \rangle + \langle S_k u\, S_m v - \chi_m, \varphi \rangle.$$

The convergences (14) and (18) imply $\chi_m = u\, S_m v$ and the following inequality holds $\|u\, S_m v\|_{p,t} \le C \|u\|_{p,s} \|v\|_{p,r}$. This estimate implies the existence of a distribution $\chi \in L^{p,t}$ such that

$$u\, S_m v \to \chi, \text{ in } L^{p,t} \text{ weakly}, \qquad \text{and} \qquad \|\chi\|_{p,t} \le C \|u\|_{p,s} \|v\|_{p,r}. \tag{19}$$

By use of (5) it follows that $\langle v, u\varphi \rangle$ and $\langle S_m v, u\varphi \rangle$ are well-defined. Moreover, due to (15) this estimate implies the strong convergence of the sequence $\varphi S_k u$ to the product φu in the space $L^{q,-r}$. Therefore by taking the limit, when k tends to infinity, in the equation $\langle S_k u\, S_m v, \varphi \rangle = \langle S_m v, \varphi S_k u \rangle$, one gets $\langle u\, S_m v, \varphi \rangle = \langle S_m v, \varphi u \rangle$. Hence

$$\langle \chi, \varphi \rangle - \langle v, u\varphi \rangle = \langle \chi - u\, S_m v, \varphi \rangle + \langle S_m v - v, u\varphi \rangle,$$

so by (19) and (16) it follows $\langle \chi, \varphi \rangle = \langle v, u\varphi \rangle$ and the distribution χ is unique. The inequality in (19) proves the theorem. ∎

ACKNOWLEDGMENTS

It is the author's pleasure to thank Professors M. T. Lacey and N. Antonić for helpful discussions.

REFERENCES

1. R. A. Adams, "Sobolev Spaces," Academic Press, New York/San Francisco/London, 1975.
2. D. R. Adams and L. I. Hedberg, "Function Spaces and Potential Theory," Springer-Verlag, Berlin/Heidelberg, 1996.
3. M. T. Lacey, Sharp estimates of the Sobolev norm of u times the gradient v, *J. Math. Anal. Appl.* **205** (1997), 554–559.
4. M. Oberguggenberger, "Multiplication of Distributions and Applications to Partial Differential Equations," Pitman Research Notes in Mathematics, Longman, Harlow, 1992.
5. E. M. Stein, "Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals," Princeton Univ. Press, Princeton, NJ, 1993.
6. R. Temam, "Navier–Stokes Equations and Nonlinear Functional Analysis," 2nd ed., CBMS-NSF Conference Series in Applied Mathematics, Vol. 66, SIAM, Philadelphia, 1995.
7. V. Girault and P-A. Raviart, "Finite Element Methods for Navier–Stokes Equations," Princeton Univ. Press, Princeton, NJ, 1970.